

Quantum hydrodynamic approximations to the finite temperature trapped Bose gases

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March 3, 2017

Abstract

For the quantum kinetic system modelling the Bose-Einstein Condensate that accounts for interactions between condensate and excited atoms, we use the Chapman-Enskog expansion to derive its hydrodynamic approximations, include both Euler and Navier-Stokes approximations. The hydrodynamic approximations describe not only the macroscopic behavior of the BEC but also its coupling with the non-condensates, which agrees with Landau's two fluid theory.

Keyword: Low and high temperature quantum kinetics; Bose-Einstein condensate; quantum Boltzmann equation; defocusing cubic nonlinear Schrodinger equation; quantum hydrodynamics limit.

MSC: 82C10, 82C22, 82C40.

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1 Introduction

After the discovery of Bose-Einstein condensations (BECs) in trapped atomic vapors of ^{87}Rb , ^7Li , and ^{23}Na [3, 4], a new period of intense experimental and theoretical research has been initiated. The equilibrium properties of these novel systems have been quite well understood, but there are still several open questions concerning their nonequilibrium behavior. One of the most important questions concerns the behavior of the condensate after cooling a nondegenerate trapped Bose gas to a temperature below the BEC critical temperature. While the experimental research has, up to now, concentrated mainly on the initial formation of BECs, their theoretical behaviour at finite temperatures is a frontier of many-body physics. The theoretical description of BECs has to take into account the coupled nonequilibrium dynamics of both the condensed and noncondensed components of the gas under investigation, and has to involve the collisional processes of atoms between the two components. Such a quantum kinetic theory was initiated by Kirkpatrick and Dorfman [32, 33]. The terminology “Quantum Kinetic Theory” has been later introduced in a series of papers by Gardinier, Zoller and collaborators [24, 22, 29, 23]. After that, there has been an explosion of research on quantum kinetic theory (see [10, 11, 12, 32, 33, 40, 25, 24, 22, 29, 23, 6, 8, 7, 2, 38, 26, 16, 35, 27, 21, 18, 44, 45], and references therein). We refer to the review paper [5] and the books [28, 36], for more discussions and a complete list of references on this rapidly expanding topic.

The current paper is devoted to the study of the hydrodynamic approximations of such a quantum kinetic system. The system contains two equations: a quantum Boltzmann equation describing the non-condensate atoms (with two types of collisions, one between excited atoms and one between condensate atoms and excited atoms), and a nonlinear Schrödinger (or Gross-Pitaevski) equation for the condensate. The hydrodynamic limits of the system is an interesting mathematical question, first studied in [1], where an Euler limit has been derived. This derivation relies on the assumption that, in the considered trapped Bose gas, the noncondensate and condensate share the same local equilibrium. It is known (cf. [32, 33]) that the condition of complete local equilibrium between the condensate and the thermal cloud requires the energy of a condensate atom in the local rest frame of the thermal cloud to be equal to the local thermal cloud chemical potential. When the condition is satisfied, there is no exchange of particles between the condensate and the thermal cloud (cf. [25]). As a consequence, in the derived fluid system, the mass of each component - condensate and non-condensate - does not exchange. Note that the two-fluid low-frequency dynamics of superfluid ^4He was first developed by Tisza and Landau [34]. Their description accounts for the characteristic features associated with superfluidity in terms of the relative motion of superfluid and normal fluid degrees of freedom, and was

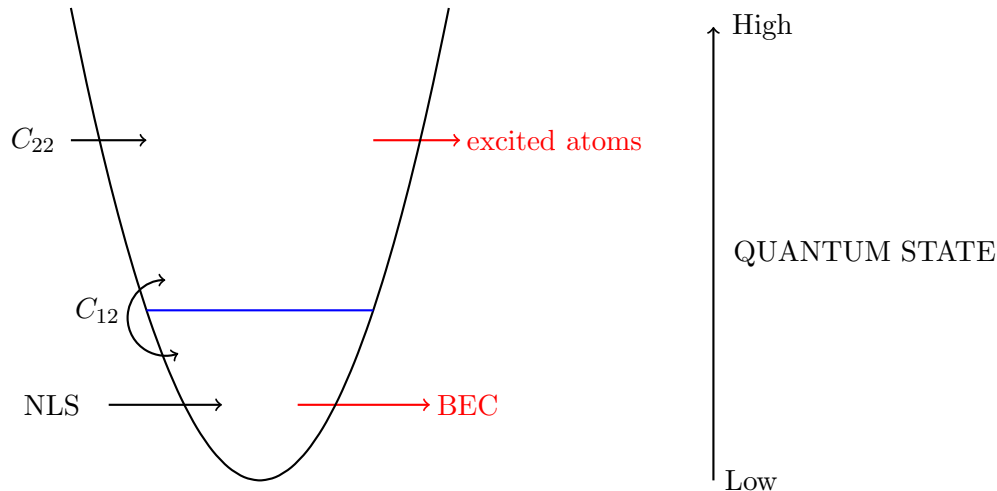


Figure 1: The Bose-Einstein Condensate (BEC) and the excitations.

shown to be a consequence of a Bose broken symmetry (cf. [13]). In the Landau two-fluid theory, the two components superfluid and normal fluid exchange mass (cf. [1, 34, 13]). In this paper, we revisit the derivation of the Euler hydrodynamic limit of the system by a different point of view: following [32, 33, 25], we assume that even if the thermal cloud atoms are in equilibrium among themselves, the noncondensate and condensate parts may not be in local equilibrium with each other. Moreover, the derivation of the Navier-Stokes approximation of the system is also provided via the classical Chapman-Enskog expansion (cf. [41]). In such circumstance, the Euler limit includes the mass exchange between the condensate and the non-condensate. Our Euler and Navier-Stokes approximations agree with the Landau two-fluid theory (cf. [34, 13]).

The plan of the paper is as follows. In Section 2 we introduce the quantum kinetic system and the scalings that will lead to the hydrodynamic approximation. In Section 3, we list the most important features of the two collision operator C_{12} and C_{22} . Section 4 is devoted to the derivation of the equation of the super-fluid of the condensate using the WKB analysis for the Gross-Pitaevski equation. The two-fluid Euler and Navier-Stokes limits are then derived in the two Sections 5 and 6 respectively.

2 The quantum kinetic system and scalings

2.1 The quantum kinetic system

Let us consider a trap Bose gas, whose temperature T is smaller than the Bose-Einstein transition temperature T_{BEC} and strictly greater than 0 K or $273.15^\circ C$. Denote $f(t, r, p)$ to be the density function of the Bose gas at time t , position r and momentum p and $\Phi(t, r)$ be the wave function of the BEC. Employing the short-handed notation $f_i = f(t, r, p_i)$, $i =$

1, 2, 3, 4, we first recall the quantum kinetic - Schrödinger system describing the dynamics of a BEC and its thermal cloud. The Schrödinger (or the Gross-Pitaevski) equation for the condensates reads (cf. [12]):

$$\begin{aligned}
i\hbar\partial_t\Phi(t, r) &= \left(-\frac{\hbar^2\Delta_r}{2m} + g[n_c(t, r) + 2n_n(t, r)] - i\Lambda_{12}[f](t, r) + V(r) \right)\Phi(t, r), \quad (t, r) \in \mathbb{R}_+ \times \mathbb{R}^3, \\
\Lambda_{12}[f](t, r) &= \frac{\hbar}{2n_c}\Gamma_{12}[f](t, r), \\
\Gamma_{12}[f](t, r) &= \int_{\mathbb{R}^3} C_{12}[f](t, r, p) \frac{dp}{(2\pi\hbar)^3}, \\
n_n(t, r) &= \int_{\mathbb{R}^3} f(t, r, p) dp, \\
\Phi(0, r) &= \Phi_0(r), \forall r \in \mathbb{R}^3,
\end{aligned} \tag{2.1}$$

where $n_c(t, r) = |\Phi|^2(t, r)$ is the condensate density, \hbar is the Planck constant, g is the interaction coupling constant proportional to the s -wave scattering length a , $V(r)$ is the confinement potential, and the operator C_{12} can be found in the quantum Boltzmann equation for the non-condensate atoms (cf. [12]), written below:

$$\partial_t f(t, r, p) + \frac{p}{m} \cdot \nabla_r f(t, r, p) - \nabla_r U(t, r) \cdot \nabla_p f(t, r, p) \tag{2.2}$$

$$= Q[f](t, r, p) := C_{12}[f](t, r, p) + C_{22}[f](t, r, p), (t, r, p) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3,$$

$$\begin{aligned}
C_{12}[f](t, r, p_1) &:= \lambda_1 n_c(t, r) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \delta(mv_c + p_1 - p_2 - p_3) \delta(\mathcal{E}_c + \mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \\
&\quad \times [(1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3)] dp_2 dp_3
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&- 2\lambda_1 n_c(t, r) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \delta(mv_c + p_2 - p_1 - p_3) \delta(\mathcal{E}_c + \mathcal{E}_{p_2} - \mathcal{E}_{p_1} - \mathcal{E}_{p_3}) \\
&\quad \times [(1 + f_2)f_1f_3 - f_2(1 + f_1)(1 + f_3)] dp_2 dp_3,
\end{aligned}$$

$$\begin{aligned}
C_{22}[f](t, r, p_1) &:= \lambda_2 \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \delta(p_1 + p_2 - p_3 - p_4) \\
&\quad \times \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&\times [(1 + f_1)(1 + f_2)f_3f_4 - f_1f_2(1 + f_3)(1 + f_4)] dp_2 dp_3 dp_4, \\
f(0, r, p) &= f_0(r, p), (r, p) \in \mathbb{R}^3 \times \mathbb{R}^3,
\end{aligned}$$

where $\lambda_1 = \frac{2g^2}{(2\pi)^2\hbar^4}$, $\lambda_2 = \frac{2g^2}{(2\pi)^5\hbar^7}$, m is the mass of the particles, \mathcal{E}_p is the Hartree-Fock energy (cf. [12])

$$\mathcal{E}_p = \mathcal{E}(p) = \frac{|p|^2}{2m} + U(t, r). \tag{2.5}$$

Notice that C_{22} is the Boltzmann-Norheim (Uehling-Uhlenbeck) quantum Boltzmann collision operator. If one writes

$$\Phi = |\Phi(t, r)| e^{i\phi(r, t)}, \tag{2.6}$$

the condensate velocity can be defined as

$$v_c(t, r) = \frac{\hbar}{m} \nabla \phi(t, r), \quad (2.7)$$

and the condensate chemical potential is then

$$\mu_c = \frac{1}{\sqrt{n_c}} \left(-\frac{\hbar^2 \Delta_r}{2m} + V + g[2n_n + n_c] \right) \sqrt{n_c}. \quad (2.8)$$

The potential U and the condensate energy \mathcal{E}_c are written as follows

$$U(t, r) = V(r) + 2g[n_c(t, r) + n_n(t, r)], \quad (2.9)$$

and

$$\mathcal{E}_c(t, r) = \mu_c(t, r) + \frac{mv_c^2(t, r)}{2}. \quad (2.10)$$

For the sake of simplicity, we suppose that $V \equiv 0$ and define the differential quantity

$$\bar{d}p = \frac{dp}{(2\pi\hbar)^3}. \quad (2.11)$$

Notice that (2.3) describes collisions between the condensate and the non-condensate atoms (condensate growth term) and (2.4) describes collisions between non-condensate atoms. Moreover, in this temperature regime, the derived collision kernels are approximated to be 1 (cf. [32, 19, 12]).

Remark 2.1 *At temperature T , bosons of mass m can be regarded as quantum-mechanical wavepackets which have an extent on the order of a thermal de Broglie wavelength $\lambda_{dB} = \left(\frac{2\pi\hbar^2}{mk_B T} \right)^{\frac{1}{2}}$, where k_B is the Boltzmann constant. The de Broglie wavelength λ_{dB} describes the position uncertainty associated with the thermal momentum distribution. When the gas temperature is high $T > T_{BEC}$, λ_{dB} is very small and the weakly interacting gas can be treated as a system of “billiard balls” (cf. [17, 31]). The dynamics of the gas is described by the Boltzmann-Norheim (Uehling-Ulenbeck) equation, whose operator sometimes reads (cf. [42])*

$$\begin{aligned} \mathcal{C}_{22}[f](t, r, p_1) &= \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \delta(p_1 + p_2 - p_3 - p_4) \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times \\ &\times [(1 + \vartheta f_1)(1 + \vartheta f_2)f_3f_4 - f_1f_2(1 + \vartheta f_3)(1 + \vartheta f_4)] dp_2 dp_3 dp_4, \end{aligned} \quad (2.12)$$

where ϑ is proportional to \hbar^3 . In the semiclassical limit, as ϑ tends to 0, the quantum Boltzmann collision operator becomes the classical one. This means at high temperature, the behavior of the “billiard balls” Bose gas is, in some sense, still very similar to classical gases.

At the BEC transition temperature, λ_{dB} becomes comparable to the distance between atoms. As a result, the atomic wavepackets “overlap” and the indistinguishability of atoms

becomes important. At this temperature, bosons undergo a quantum-mechanical phase transition and the Bose-Einstein condensate is formed (cf. [17, 31]). When the temperature of the gas is finite $T_{BEC} > T > 0K$, the trapped Bose gas is composed of two distinct components: the high-density condensate, being localized at the center of the trapping potential, and the low-density cloud of thermally excited atoms, spreading over a much wider region. The dynamics of the thermal cloud atoms is described by the kinetic equation (2.2). At this low temperature, the de Broglie wavelength of the excited atoms is very large, in comparison with the high temperature boson de Broglie wavelength. As a consequence, the thermal cloud kinetic equation cannot be treated as a system of “billiard balls” anymore. This explains the difference between the forms of the two collision operators C_{22} and \tilde{C}_{22} .

Note that, different from classical Boltzmann collision operators, where the collision kernels are functions depending on the types of particles considered, the derived collision kernel for the quantum Boltzmann collision operator for bosons is 1 (cf. [20]) when $T > T_{BEC}$.

2.2 Scalings

Different from the previous work [1], in which the two collision operators C_{12} and C_{22} are assumed to have the same equilibrium distribution function, we follow [25] to consider a different regime, where excited atoms in the condensate need not to be in local equilibrium with the condensate atoms. As a consequence, C_{12} and C_{22} in general do not share the same equilibrium distribution. Relying on these physical assumptions, we propose a new approach based on selecting a small parameter ϵ , to be used for the Chapman-Enskog expansion. Let us describe the choice of this small parameter below.

It is known that the dynamics of the trapped Bose gases depends on its temperature T . Let us restrict our attention to the case where T is smaller but very close to the Bose-Einstein critical temperature T_{BEC} . At this temperature regime, the collisions between excited atoms are very rapid to establish a local equilibrium within the non-condensate component. As a consequence, the collision operator C_{22} is considered to be much stronger than the collision operator C_{12} . This regime is often called *the state of partial local equilibrium* which arises near T_{BEC} when the density of the condensate is small.

Following (cf. [25]), we define the *static equilibrium* of the system

$$\mathcal{F}_0(p) = \frac{1}{e^{\beta_0[(p-mv_{n0})^2/(2m)+U_0-\mu_0]} - 1}, \quad (2.13)$$

where β_0 is the static temperature parameter, $v_{n0}(t, r)$ is the static fluid velocity, $\mu_0(t, r)$ is the static chemical potential, U_0 is the static mean field. We also set the static density to be

$$n_{n0} = \int_{\mathbb{R}^3} \mathcal{F}_0(p) \bar{d}p. \quad (2.14)$$

Note that when T is sufficiently close to T_{BEC} , the bosons are in the *particle-like regime*, i.e. they behave like particles. Let us also mention that when temperature T is very close to 0, the bosons will be in the phonon-like regime (cf. [37]). Since we are interested in the

behavior of the particles when T is close to T_{BEC} , let us define the collision frequency with respect to C_{12}

$$\begin{aligned}\nu_{12}(p_1) &= \frac{\lambda_1 n_c}{m^2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \delta(mv_c + p_1 - p_2 - p_3) \delta(\mathcal{E}_c + \mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \mathcal{F}_0(p_2) \mathcal{F}_0(p_3) dp_2 dp_3 \\ &\quad + 2 \frac{\lambda_1 n_c}{m^2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \delta(mv_c + p_2 - p_1 - p_3) \delta(\mathcal{E}_c + \mathcal{E}_{p_2} - \mathcal{E}_{p_1} - \mathcal{E}_{p_3}) [\mathcal{F}_0(p_2)(\mathcal{F}_0(p_1) \\ &\quad + \mathcal{F}_0(p_2) + 1)] dp_2 dp_3,\end{aligned}\tag{2.15}$$

as well as the associated mean collision frequency:

$$\bar{\nu}_{12} = \frac{1}{n_{n0} m^2} \int_{\mathbb{R}^3} \nu_{12}(p) \mathcal{F}_0(p) \bar{d}p.\tag{2.16}$$

The inverse of $\nu_{12}(p)$ and $\bar{\nu}_{12}$ are defined to be, respectively, the free time $\tau_{12}(p)$ and the mean field time $\bar{\tau}_{12}$:

$$\tau_{12}(p) = \frac{1}{\nu_{12}(p)}, \quad \bar{\tau}_{12} = \frac{1}{\bar{\nu}_{12}}.\tag{2.17}$$

We now determine the average speed of the particles

$$\bar{c} = \frac{1}{n_{n0} m} \int_{\mathbb{R}^3} \sqrt{p^2} \mathcal{F}_0(p) \bar{d}p,\tag{2.18}$$

and the mean free path

$$l_{12} = \bar{c} \bar{\tau}_{12}.\tag{2.19}$$

Similarly, the collision frequency and the mean collision frequency associated to C_{22} can be defined

$$\begin{aligned}\nu_{22}(p_1) &= \frac{\lambda_2}{m^2} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \delta(p_1 + p_2 - p_3 - p_4) \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times \\ &\quad \times \mathcal{F}_0(p_1) \mathcal{F}_0(p_2) (1 + \mathcal{F}_0(p_3)) (1 + \mathcal{F}_0(p_4)) dp_2 dp_3 dp_4,\end{aligned}\tag{2.20}$$

and

$$\bar{\nu}_{22} = \frac{1}{n_{n0} m^2} \int_{\mathbb{R}^3} \nu_{22}(p) \mathcal{F}_0(p) \bar{d}p.\tag{2.21}$$

We also define the free time $\tau_{22}(p)$, the mean field time $\bar{\tau}_{22}$ and the mean free path l_{22}

$$\tau_{22}(p) = \frac{1}{\nu_{22}(p)}, \quad \bar{\tau}_{22} = \frac{1}{\bar{\nu}_{22}}, \quad l_{22} = \bar{c} \bar{\tau}_{22}.\tag{2.22}$$

Let L and θ be the reference length and time, respectively. Following [39, 14], we introduce the rescaled variables

$$\tilde{r} = \frac{r}{L}, \quad \tilde{t} = \frac{t}{\theta}, \quad \tilde{p} = \frac{p}{P}, \quad P = m\bar{c}.\tag{2.23}$$

We also rescale U as $\tilde{U} = U/U_0$, where U_0 is the reference potential field. Define

$$\begin{aligned}\tilde{C}_{12}[f](t, r, \tilde{p}_1) &:= \tilde{\lambda}_1 n_c(t, r) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \delta(mv_c + \tilde{p}_1 - \tilde{p}_2 - \tilde{p}_3) \delta(\mathcal{E}_c + \mathcal{E}_{\tilde{p}_1} - \mathcal{E}_{\tilde{p}_2} - \mathcal{E}_{\tilde{p}_3}) \\ &\quad \times [(1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3)] d\tilde{p}_2 d\tilde{p}_3 \\ &\quad - 2\tilde{\lambda}_1 n_c(t, r) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \delta(mv_c + \tilde{p}_2 - \tilde{p}_1 - \tilde{p}_3) \delta(\mathcal{E}_c + \mathcal{E}_{\tilde{p}_2} - \mathcal{E}_{\tilde{p}_1} - \mathcal{E}_{\tilde{p}_3}) \\ &\quad \times [(1 + f_2)f_1f_3 - f_2(1 + f_1)(1 + f_3)] d\tilde{p}_2 d\tilde{p}_3,\end{aligned}\tag{2.24}$$

$$\begin{aligned}\tilde{C}_{22}[f](t, r, \tilde{p}_1) &:= \tilde{\lambda}_2 \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \delta(\tilde{p}_1 + \tilde{p}_2 - \tilde{p}_3 - \tilde{p}_4) \\ &\quad \times \delta(\mathcal{E}_{\tilde{p}_1} + \mathcal{E}_{\tilde{p}_2} - \mathcal{E}_{\tilde{p}_3} - \mathcal{E}_{\tilde{p}_4}) \times \\ &\quad \times [(1 + f_1)(1 + f_2)f_3f_4 - f_1f_2(1 + f_3)(1 + f_4)] d\tilde{p}_2 d\tilde{p}_3 d\tilde{p}_4,\end{aligned}\tag{2.25}$$

where

$$\tilde{\lambda}_{12} = P^2 \lambda_{12},\tag{2.26}$$

and

$$\tilde{\lambda}_{22} = P^3 \lambda_{22}.\tag{2.27}$$

As a consequence, we can define the rescaled mean free paths and the rescaled mean field times to be

$$\tilde{l}_{22} = \frac{l_{22}}{P^3}, \quad \tilde{\tau}_{22} = P^3 \tau_{22},$$

and

$$\tilde{l}_{12} = \frac{l_{12}}{P^2}, \quad \tilde{\tau}_{12} = P^2 \tau_{12}.$$

We also set

$$\hat{C}_{12}[f] := \tilde{l}_{12} \tilde{C}_{12}[f], \quad \hat{C}_{22}[f] := \tilde{l}_{22} \tilde{C}_{22}[f].\tag{2.28}$$

The following rescaled version of (2.2) then follows:

$$\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{\theta\tilde{c}} \partial_{\tilde{t}} f + \frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{L} \frac{P}{m\tilde{c}} \tilde{p} \cdot \nabla_{\tilde{r}} f - \frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{L} \frac{U_0}{P\tilde{c}} \nabla_{\tilde{r}} \tilde{U} \cdot \nabla_{\tilde{p}} f = \sqrt{\frac{\tilde{l}_{22}}{\tilde{l}_{12}}} \hat{C}_{12}[f] + \sqrt{\frac{\tilde{l}_{12}}{\tilde{l}_{22}}} \hat{C}_{22}[f].\tag{2.29}$$

Notice that $\frac{\tilde{\tau}_{22}}{\tilde{\tau}_{12}} = \frac{\tilde{l}_{22}}{\tilde{l}_{12}}$ is a dimensionless parameter and is proportional to $\frac{\tilde{\lambda}_1}{\tilde{\lambda}_2}$. Since in the temperature regime under consideration, the collisions between excited atoms are very rapid to establish a local equilibrium within the non-condensate component, we now make the assumption that $\tilde{\tau}_{22} \ll \tilde{\tau}_{12}$ and suppose $\frac{\tilde{\tau}_{22}}{\tilde{\tau}_{12}} = \epsilon^2$. The constants $\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{\theta\tilde{c}}$, $\frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{L}$ can be set to be 1 by rescaling again the space and time variables $\tilde{t} \rightarrow \frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{\theta\tilde{c}} \tilde{t}$, $\tilde{r} \rightarrow \frac{\sqrt{\tilde{l}_{12}\tilde{l}_{22}}}{L} \tilde{r}$, and note that $\frac{P}{m\tilde{c}} = 1$, we obtain the following equation

$$\partial_{\tilde{t}} f + \tilde{p} \cdot \nabla_{\tilde{r}} f - \frac{U_0}{m\tilde{c}^2} \nabla_{\tilde{r}} \tilde{U} \cdot \nabla_{\tilde{p}} f = \epsilon \hat{C}_{12}[f] + \frac{1}{\epsilon} \hat{C}_{22}[f].\tag{2.30}$$

Notice that g is also the principle small parameter used in the derivation of the system (2.1)-(2.2). Indeed, the derivation starts with the usual Heisenberg equation of motion for the quantum field operator. The equation for the condensate wavefunction follows by averaging the Heisenberg equation with respect to a broken-symmetry nonequilibrium ensemble. Taking the difference between the Heisenberg equation and the equation for the condensate wavefunction and keeping only the terms of low orders with respect to g , we obtain the equation of the noncondensate field operator, which, by a Wigner transform, leads to the Wigner equation. In this process, one computes the collision integrals C_{12} , C_{22} to second order $O(g^2)$ in g and keep interaction effects in the excitation energies and chemical potential only to first order $O(g)$. For a more detailed explanation of this procedure, we refer to, for instance, Sections 3.1, 3.2 and 5.3 of the book [25]. Since U_0 has to be chosen proportional to g , the dimensionless parameter $\frac{U_0}{mc^2}$ might be considered to be small and set it to be $\tilde{g} = \epsilon^{\delta_0}$, $0 < \delta_0 < 1$. *In Section 5, the difference between the hydrodynamics limits for the two choices of the magnitude of \tilde{g} , small $O(\epsilon^{\delta_0})$ and not small $O(1)$, will be compared.*

The equation then follows, as a result of the previous scaling

$$\partial_t f + \tilde{p} \cdot \nabla_{\tilde{r}} f - \tilde{g} \nabla_{\tilde{r}} \tilde{U} \cdot \nabla_{\tilde{p}} f = \epsilon \hat{C}_{12}[f] + \frac{1}{\epsilon} \hat{C}_{22}[f]. \quad (2.31)$$

Under this scaling, the Gross-Pitaevski equation also becomes

$$i \frac{\hbar}{\theta} \partial_t \Phi(t, r) = \left(-\frac{\hbar^2 \Delta_{\tilde{r}}}{2mL^2} + g[n_c(t, r) + 2n_n(t, r)] - \frac{i\hbar}{\tau_{12}} \tilde{\Lambda}_{12}[f](t, r) \right) \Phi(t, r), \quad (2.32)$$

where

$$\tilde{\Lambda}_{12}[f] = \frac{1}{2n_c} \int_{\mathbb{R}^3} \hat{C}_{12}[f] \tilde{d}p.$$

By the same argument as above, we also obtain

$$i \frac{\sqrt{\tilde{l}_{12} \tilde{l}_{22}}}{\theta \tilde{c}} \partial_t \Phi(t, r) = \left(-\frac{\hbar}{mL} \frac{\sqrt{\tilde{l}_{12} \tilde{l}_{22}} \Delta_{\tilde{r}}}{2L\tilde{c}} + \frac{\sqrt{\tilde{l}_{12} \tilde{l}_{22}}}{\hbar \tilde{c}} U_*(t, r) - \frac{i\sqrt{\tilde{l}_{12} \tilde{l}_{22}}}{\tilde{l}_{12}} \tilde{\Lambda}_{12}[f](t, r) \right) \Phi(t, r), \quad (2.33)$$

where $U_*(t, r) = g[n_c(t, r) + 2n_n(t, r)]$. Since $U_*(t, r)$ has the dimension of $U(t, r)$, we can rescale it as $\tilde{U}_* = U_*/U_0$, to get

$$i \frac{\sqrt{\tilde{l}_{12} \tilde{l}_{22}}}{\theta \tilde{c}} \partial_t \Phi(t, r) = \left(-\frac{\hbar}{mL} \frac{\sqrt{\tilde{l}_{12} \tilde{l}_{22}} \Delta_{\tilde{r}}}{2L\tilde{c}} + \frac{\sqrt{\tilde{l}_{12} \tilde{l}_{22}} U_0}{\hbar \tilde{c}} \tilde{U}_*(t, r) - \frac{i\sqrt{\tilde{l}_{12} \tilde{l}_{22}}}{\tilde{l}_{12}} \tilde{\Lambda}_{12}[f](t, r) \right) \Phi(t, r), \quad (2.34)$$

Notice that $\frac{\hbar}{m\tilde{c}}$ has the dimensions of a length (Compton wavelength) and $\frac{\hbar}{mL\tilde{c}}$ is dimensionless; hence the quantity $\frac{\hbar}{mL} \frac{\sqrt{\tilde{l}_{12} \tilde{l}_{22}}}{2L\tilde{c}}$ is dimensionless. Moreover, $\frac{\sqrt{\tilde{l}_{12} \tilde{l}_{22}} U_0}{\hbar \tilde{c}}$ is the product of the three dimensionless parameters $\frac{\sqrt{\tilde{l}_{12} \tilde{l}_{22}}}{L}$, $\frac{mL\tilde{c}}{\hbar}$ and $\frac{U_0}{mc^2} = \tilde{g}$. Setting all of the dimensionless parameter to be 1 by the same rescaling argument used for (2.30) and dropping the

tilde and hat signs

$$i\partial_t\Phi(t, r) = \left(-\frac{\Delta_r}{2} + gU_*(t, r) - \Lambda_{12}[f](t, r) \right) \Phi(t, r), \quad (2.35)$$

where g stands for the dimensionless parameter $\tilde{g} = \epsilon^{\delta_0}$, we then obtain the system

$$\begin{aligned} \partial_t f + p \cdot \nabla_r f - g \nabla_r U \cdot \nabla_p f &= \epsilon C_{12}[f] + \frac{1}{\epsilon} C_{22}[f], \quad (0 < \delta_0 < 1), \\ i\partial_t \Phi(t, r) &= \left(-\frac{\Delta_r}{2} + g[n_c(t, r) + 2n_n(t, r)] - i\epsilon \Lambda_{12}[g](t, r) \right) \Phi(t, r). \end{aligned} \quad (2.36)$$

We recall below the formulas for C_{12} , C_{22} and Λ_{12}

$$\begin{aligned} C_{12}[f](t, r, p_1) &= n_c(t, r) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \delta(mv_c + p_1 - p_2 - p_3) \delta(\mathcal{E}_c + \mathcal{E}_{\tilde{p}_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) \\ &\quad \times [(1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3)] dp_2 dp_3 \\ &\quad - 2n_c(t, r) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \delta(mv_c + p_2 - p_1 - p_3) \delta(\mathcal{E}_c + \mathcal{E}_{p_2} - \mathcal{E}_{p_1} - \mathcal{E}_{p_3}) \\ &\quad \times [(1 + f_2)f_1f_3 - f_2(1 + f_1)(1 + f_3)] dp_2 dp_3, \end{aligned} \quad (2.37)$$

$$\begin{aligned} C_{22}[f](t, r, p_1) &= \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \delta(p_1 + p_2 - p_3 - p_4) \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times \\ &\quad \times [(1 + f_1)(1 + f_2)f_3f_4 - f_1f_2(1 + f_3)(1 + f_4)] dp_2 dp_3 dp_4, \end{aligned} \quad (2.38)$$

$$\Lambda_{12}[f](t, r) = \frac{1}{n_c(t, r)} \int_{\mathbb{R}^3} C_{12}[f](t, r, p) dp. \quad (2.39)$$

We also define the differential operators

$$\mathcal{D}f = \partial_t f + \frac{p}{m} \cdot \nabla_r f - g \nabla_r U(t, r) \cdot \nabla_p f - C_{12}[f], \quad (2.40)$$

$$\mathbb{D}f = \partial_t f + p \cdot \nabla_r f - g \nabla_r U(t, r) \cdot \nabla_p f, \quad (2.41)$$

$$\Pi f = \partial_t f + p \cdot \nabla_r f, \quad (2.42)$$

and then get

$$\mathbb{D}f = \epsilon C_{12}[f] + \frac{1}{\epsilon} C_{22}[f], \quad (0 < \delta_0 < 1), \quad (2.43)$$

$$i\partial_t \Phi(t, r) = \left(-\frac{\Delta_r}{2} + g[n_c(t, r) + 2n_n(t, r)] - i\epsilon \Lambda_{12}[g](t, r) \right) \Phi(t, r). \quad (2.44)$$

The new constant ϵ is the small parameter that we will use in the usual Chapman-Enskog expansion process.

3 Properties of the collision operators

In this section, we study the main properties of the two collision operators C_{12} and C_{22} .

Let us start with C_{22} , which can be represented as:

$$C_{22}[f] = B_1[f, f] + B_2[f, f, f], \quad (3.1)$$

in which

$$\begin{aligned} B_1[f, g] &= \frac{1}{2} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \delta(p_1 + p_2 - p_3 - p_4) \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times \\ &\times [f_3 g_4 + f_4 g_3 - f_1 g_2 - f_2 g_1] dp_2 dp_3 dp_4, \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} B_2[f, g, h] &= \frac{1}{6} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \delta(p_1 + p_2 - p_3 - p_4) \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \times \\ &\times [f_3 g_4 h_1 + f_4 g_3 h_1 + f_3 g_4 h_2 + f_4 g_3 h_2 \\ &\quad + f_1 g_4 h_3 + f_1 g_3 h_4 + f_2 g_4 h_3 + f_3 g_3 h_4 \\ &\quad + f_4 g_1 h_3 + f_3 g_1 h_4 + f_4 g_2 h_3 + f_3 g_2 h_4 \\ &\quad - f_1 g_2 h_3 - f_2 g_1 h_3 - f_1 g_2 h_4 - f_2 g_1 h_4 \\ &\quad - f_3 g_1 h_2 - f_3 g_2 h_1 - f_4 g_1 h_2 - f_4 g_2 h_1 \\ &\quad - f_1 g_3 h_2 - f_2 g_3 h_1 - f_1 g_4 h_2 - f_2 g_4 h_1] dp_2 dp_3 dp_4, \end{aligned} \quad (3.3)$$

where we have used the same notations $f_1, f_2, f_3, f_4, g_1, g_2, g_3, g_4$ with the ones used in (2.2).

The operator C_{22} shares some important features with the classical Boltzmann collision operator. Among these features, the following can be proved by switching the variables $p_1 \leftrightarrow p_2, p_1 \leftrightarrow p_3$ in the integrals of I_1 and $(p_1, p_2) \leftrightarrow (p_2, p_1), (p_1, p_2) \leftrightarrow (p_3, p_4)$, respectively, as in the classical case (cf. [43]):

$$\int_{\mathbb{R}^3} \Psi_i(p) B_1[f, f](p) dp = 0, \quad i = 0, 1, 2, 3, 4, \quad (3.4)$$

and

$$\int_{\mathbb{R}^3} \Psi_i(p) B_2[f, f, f](p) dp = 0, \quad i = 0, 1, 2, 3, 4, \quad (3.5)$$

where

$$\Psi_0(p) = 1, \quad \Psi_i(p) = p_i, \quad (i = 1, 2, 3), \quad \Psi_4(p) = |p|^2, \quad (3.6)$$

are the collision invariants and p_i is the i -th component of the vector p .

Similar as the classical Boltzmann collision operator, C_{22} also has a local equilibrium of the form

$$\mathcal{F}(t, r, p) = \frac{1}{e^{\beta[(p-v_n)^2 + U - \mu]} - 1}, \quad (3.7)$$

where $\beta(t, r)$ is the temperature parameter, $v_n(t, r)$ is the local fluid velocity, $\mu(t, r)$ is the local chemical potential (which is different from the condensate chemical potential $\mu_c(t, r)$ defined in (2.8)), $U(t, r)$ is the mean field. Then

$$C_{22}[\mathcal{F}] = 0.$$

Let us now define the following Gaussian

$$\mathcal{M}(t, r, p) = \gamma(t, r) e^{-\frac{|p - u(t, r)|^2}{2\tau(t, r)}}, \quad (3.8)$$

where

$$\gamma(t, r) = e^{\beta(U(t, r) - \mu(t, r))}, \quad u(t, r) = v_n(t, r), \quad \tau(t, r) = \frac{1}{\beta(t, r)}. \quad (3.9)$$

The local equilibrium \mathcal{F} can be expressed in terms of \mathcal{M} as

$$\mathcal{F}(t, r, p) = \frac{\mathcal{M}(t, r, p)}{1 - \mathcal{M}(t, r, p)}. \quad (3.10)$$

Let $L^2(\mathbb{R}^3)$ be the space of real, measurable functions, whose second power is integrable on \mathbb{R}^3 , with the norm $\|\cdot\|_{L^2}$ and inner product $(\cdot, \cdot)_{L^2}$. We consider the linearized operator of C_{22} around a fixed equilibrium $\mathcal{F}(t, r, p)$, which, by a classical process can be defined as

$$\mathcal{L} := 2B_1(\mathcal{F}, \cdot) + 3B_2(\mathcal{F}, \mathcal{F}, \cdot), \quad (3.11)$$

or equivalently

$$\begin{aligned} \mathcal{L}(\mathcal{F}f)(t, r, p_1) &= \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \delta(p_1 + p_2 - p_3 - p_4) \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \\ &\quad \times \frac{\mathcal{M}_1 \mathcal{M}_2}{(1 - \mathcal{M}_1)(1 - \mathcal{M}_2)(1 - \mathcal{M}_3)(1 - \mathcal{M}_4)} \times \\ &\quad \times \left[(1 - \mathcal{M}_3)f(p_3) + (1 - \mathcal{M}_4)f(p_4) - (1 - \mathcal{M}_2)f(p_2) \right. \\ &\quad \left. - (1 - \mathcal{M}_1)f(p_1) \right] dp_2 dp_3 dp_4, \end{aligned} \quad (3.12)$$

for some function $f(p)$ and fixed values $(t, r) \in \mathbb{R}_+ \times \mathbb{R}^3$ and we employ the shorthand notations $\mathcal{M}_i = \mathcal{M}(t, r, p_i)$, $i = 0, 1, 2, 3, 4$. Now, let us consider the inner product between the above linearized operator and some test function φ . The classical argument (cf. [43]) for the classical linearized Boltzmann collision operator can be applied and gives:

$$\begin{aligned} \left(\frac{\mathcal{M}}{\mathcal{F}} \varphi, \mathcal{L}(\mathcal{F}f) \right)_{L^2} &= \frac{1}{4} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \delta(p_1 + p_2 - p_3 - p_4) \delta(\mathcal{E}_{p_1} + \mathcal{E}_{p_2} - \mathcal{E}_{p_3} - \mathcal{E}_{p_4}) \\ &\quad \times \frac{\mathcal{M}_1 \mathcal{M}_2}{(1 - \mathcal{M}_1)(1 - \mathcal{M}_2)(1 - \mathcal{M}_3)(1 - \mathcal{M}_4)} \times \\ &\quad \times \left[(1 - \mathcal{M}_3)f(p_3) + (1 - \mathcal{M}_4)f(p_4) - (1 - \mathcal{M}_2)f(p_2) \right. \\ &\quad \left. - (1 - \mathcal{M}_1)f(p_1) \right] \left[(1 - \mathcal{M}_3)\varphi(p_3) + (1 - \mathcal{M}_4)\varphi(p_4) \right. \\ &\quad \left. - (1 - \mathcal{M}_2)\varphi(p_2) - (1 - \mathcal{M}_1)\varphi(p_1) \right] dp_2 dp_3 dp_4, \end{aligned}$$

which implies

$$\left(\frac{\mathcal{M}}{\mathcal{F}} f, \mathcal{L}(\mathcal{F}f) \right)_{L^2} \leq 0, \quad (3.13)$$

and

$$\left(\frac{\mathcal{M}}{\mathcal{F}} \varphi, \mathcal{L}(\mathcal{F}f) \right)_{L^2} = \left(\frac{\mathcal{M}}{\mathcal{F}} f, \mathcal{L}(\mathcal{F}\varphi) \right)_{L^2},$$

for all function φ and f such that the integrals are well-defined. The equality in (3.13) holds true if and only if $\frac{\mathcal{M}}{\mathcal{F}} f$ is identical to one of the five functions defined in (3.6).

From the above observation, we are now able to define the kernel of the linearized collision operator \mathcal{L} of C_{22} :

$$\mathcal{N} := \ker \mathcal{L} = \text{span} \left\{ \frac{\mathcal{F}^2}{\mathcal{M}} \Psi_i : i = 0, \dots, 4 \right\},$$

and its orthogonal space:

$$\mathcal{R} := \mathcal{N}^\perp = \left\{ G \in L^2(\mathbb{R}^3) : \left(G, \frac{\mathcal{F}^2}{\mathcal{M}} \Psi_i \right)_{L^2} = 0, i = 0, \dots, 4 \right\}.$$

On $L^2(\mathbb{R}^3)$, we also define the orthogonal projection operators \mathbb{P} and $\mathbb{P}^\perp = 1 - \mathbb{P}$ on to \mathcal{N} and \mathcal{R} . By normalizing $\{\Psi_i\}_{i=0,\dots,4}$, we obtain the following orthonormal basis of the space \mathcal{N}

$$\left\{ \frac{\psi_i}{\sqrt{\omega_i}} \frac{\mathcal{F}^2}{\mathcal{M}} : i = 0, \dots, 4 \right\}, \quad (3.14)$$

with

$$\begin{aligned} \psi_0 &= 1; \quad \psi_i = p_i - u_i, \quad i = 1, 2, 3; \quad \psi_4 = |p - u|^2 - 6\tau \frac{\Omega_1(\gamma)}{\Omega_0(\gamma)}, \\ \psi_0 &= \int_{\mathbb{R}^3} \frac{\mathcal{F}^2}{\mathcal{M}} dp = 2^{3/2} \pi \tau^{3/2} \gamma \Omega_0(\gamma); \\ \psi_i &= \int_{\mathbb{R}^3} \frac{\mathcal{F}^2}{\mathcal{M}} |\psi_i(p)|^2 dp = 2^{5/2} \pi \tau^{5/2} \gamma \Omega_1(\gamma), \quad i = 1, 2, 3; \\ \psi_i &= \int_{\mathbb{R}^3} \frac{\mathcal{F}^2}{\mathcal{M}} |\psi_4(p)|^2 dp = 2^{7/2} \pi \tau^{7/2} \gamma \Sigma(\Omega_0(\gamma), \Omega_1(\gamma), \Omega_2(\gamma)); \end{aligned}$$

where

$$\Sigma(x, y, z) = \frac{5xz - 9y^2}{x},$$

and

$$\Omega_k(\gamma) = \int_0^\infty \frac{y^{k-1/2}}{e^y - \gamma} dy, \quad k > -1/2. \quad (3.15)$$

Following the classical argument for the classical Boltzmann collision operator, define

$$\mathcal{A}(p) = p \otimes p - \frac{1}{3} |p|^2 Id, \quad \mathcal{B}(p) = \frac{1}{2} p (|p|^2 - 5). \quad (3.16)$$

By the same algebraic argument as the one used for the classical Boltzmann collision operator (cf. page 64-65 [14]), one can deduce that there exists scalar-valued functions $\alpha_0(|p|)$, $\alpha_1(|p|)$ such that

$$\mathcal{L}^{-1} \left(\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)} \mathcal{A}(p) \right) = \alpha_0(|p|) \frac{\mathcal{F}^2(p)}{\mathcal{M}(p)} \mathcal{A}(p), \quad \mathcal{L}^{-1} \left(\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)} \mathcal{B}(p) \right) = \alpha_1(|p|) \frac{\mathcal{F}^2(p)}{\mathcal{M}(p)} \mathcal{B}(p). \quad (3.17)$$

A direct consequence of (3.17) is the existence of scalar-valued functions $\beta_0(|p|)$ and $\beta_1(|p|)$ such that

$$\begin{aligned} \mathcal{L}^{-1} \left(\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)} p_i p_j \right) &= \beta_0(|p|) \frac{\mathcal{F}^2(p)}{\mathcal{M}(p)} \mathcal{A}_{ij}(p), \\ \mathcal{L}^{-1} \left(\frac{\mathcal{F}^2(p)}{\mathcal{M}(p)} \left(|p|^2 - \frac{10\tau\Omega_2(\gamma)}{\Omega_1(\gamma)} p_i \right) p_i \right) &= \beta_1(|p|) \frac{\mathcal{F}^2(p)}{\mathcal{M}(p)} \mathcal{B}_i(p), \end{aligned} \quad (3.18)$$

where p_i , $\mathcal{B}_i(p)$ are the i -th component of the vectors p and $\mathcal{B}(p)$ respectively. In addition, $\mathcal{A}_{ij}(p)$ is the (i, j) -th element of the matrix $\mathcal{A}(p)$. Note that these symmetry invariances are very similar to the ones obtained in the context of the classical Boltzmann collision operator (cf. Equation (2.100), page 64-65 [14]); we then denote

$$\mathfrak{C}_{ij}(p) := \beta_0(|p|) \frac{\mathcal{F}^2(p)}{\mathcal{M}(p)} \mathcal{A}_{ij}(p), \quad \mathfrak{C}_i(p) := \beta_1(|p|) \frac{\mathcal{F}^2(p)}{\mathcal{M}(p)} \mathcal{B}_i(p). \quad (3.19)$$

In order to study the hydrodynamics limit of the system, let us define the following moments of the function f :

$$n_n[f](t, r) = \int_{\mathbb{R}^3} f(t, r, p) dp, \quad (3.20)$$

$$v_n(t, r)[f](t, r) = \frac{1}{n_n[f](t, r)} \int_{\mathbb{R}^3} p f(t, r, p) dp, \quad (3.21)$$

$$\mathbb{E}_n[f](t, r) = \frac{1}{2} \int_{\mathbb{R}^3} f(t, r, p) |p - v_n[f](t, r)|^2 dp, \quad (3.22)$$

$$e_n[f](t, r) = \frac{\mathbb{E}_n[f](t, r)}{n_n[f](t, r)}. \quad (3.23)$$

Replacing f by \mathcal{F} , we obtain

$$\begin{aligned} n_n[\mathcal{F}] &= 2^{5/2} \pi \tau^{3/2} \Omega_1(\gamma), \\ v_n(t, r)[\mathcal{F}] &= u, \\ \mathbb{E}_n[\mathcal{F}] &= 2^{5/2} \pi \tau^{5/2} \gamma \Omega_2(\gamma). \end{aligned} \quad (3.24)$$

For the sake of simplicity, we denote $n_n[\mathcal{F}]$, $v_n(t, r)[\mathcal{F}]$, $\mathbb{E}_n[\mathcal{F}]$, $e_n[\mathcal{F}]$ by n_n , v_n , \mathbb{E}_n , and e_n .

We indeed can compute γ and τ as

$$\gamma = \left(\frac{\text{Id}\Omega_2}{\Omega_1^{5/3}} \right)^{-1} \left(\frac{2^{5/3}\pi^{2/3}\mathbb{E}_n}{n_n^{5/3}} \right), \quad (3.25)$$

and

$$\tau = \left(\frac{n_n}{2^{5/2}\pi\Omega_1 \left(\left(\frac{\text{Id}\Omega_2}{\Omega_1^{5/3}} \right)^{-1} \left(\frac{2^{5/3}\pi^{2/3}\mathbb{E}_n}{n_n^{5/3}} \right) \right)} \right)^{2/3}. \quad (3.26)$$

Now, let us consider the collision operator C_{12} . This operator also has the collision invariant property:

$$\int_{\mathbb{R}^3} (\Psi_i(p) - v_{ci}) C_{12}[f] dp = \int_{\mathbb{R}^3} (\Psi_4(p) + 2U - 2\mu_c - v_c^2) C_{12}[f] dp = 0, \quad i = 1, 2, 3. \quad (3.27)$$

An important property of C_{12} is that \mathcal{F} is not an equilibrium of C_{12} . We have:

$$\begin{aligned} \Gamma_{12}[\mathcal{F}] &:= \int_{\mathbb{R}^3} C_{12}[\mathcal{F}] dp = \\ &= -n_c [1 - e^{-\beta(\mu - \mu_c - (v_n - v_c)^2/2)}] \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \delta(v_c + p_1 - p_2 - p_3) \times \\ &\quad \times \delta(\mathcal{E}_c + \mathcal{E}_{p_1} - \mathcal{E}_{p_2} - \mathcal{E}_{p_3}) (1 + \mathcal{F}(t, r, p_1)) \mathcal{F}(t, r, p_2) \mathcal{F}(t, r, p_3) dp_1 dp_2 dp_3. \end{aligned} \quad (3.28)$$

Expanding \mathcal{F} into Taylor series of \mathcal{M} , we can simplify the above integral as

$$\begin{aligned} \Gamma_{12}[\mathcal{F}] &= \\ &= -n_c [1 - e^{-\beta(\mu - \mu_c - (v_n - v_c)^2/2)}] \sum_{k_2, k_3 \in \mathbb{N} \cup \{0\}, k_1 \in \mathbb{N}} \gamma^3 e^{-\frac{|v_c - u|^2 (k_1 + k_2 + k_3)}{2\tau}} \times \\ &\quad \times e^{\frac{(-2\mathcal{E}_c + 2U + v_c^2)k_1}{2\tau}} \int_{x \cdot y = \frac{v_c^2}{2} + U - \mathcal{E}_c} e^{-(k_1 + k_2)[|x|^2 + x \cdot (v_c - u)] - (k_1 + k_3)[|y|^2 + y \cdot (v_c - u)] / (2\tau)} dx dy, \end{aligned} \quad (3.29)$$

with the notice that from (3.25) and (3.26) γ and τ are functions of n_n and \mathbb{E}_n .

4 The Bose-Einstein condensate super-fluid

This Section is devoted to the derivation of the equation that describes the super-fluid of the Bose-Einstein Condensate. First, let us consider the nonlinear Schrödinger equation (2.44). Since

$$\Phi = |\Phi(t, r)| e^{i\phi(r, t)} = \sqrt{n_c(t, r)} e^{i\phi(r, t)},$$

one can rewrite (2.44) under the following form

$$i\partial_t \left(\sqrt{n_c} e^{i\phi} \right) = \left(-\frac{\Delta_r}{2} + g[2n_n + n_c] \right) \sqrt{n_c} e^{i\phi} - \frac{i\epsilon}{2\sqrt{n_c}} \Gamma_{12}[f] e^{i\phi}. \quad (4.1)$$

Expanding the derivative in t on the right hand side of (4.1), one finds

$$\begin{aligned} i\partial_t \left(\sqrt{n_c} e^{i\phi} \right) &= i \frac{\partial_t n_c}{2\sqrt{n_c}} e^{i\phi} + i \sqrt{n_c} e^{i\phi} i\partial_t \phi \\ &= i \frac{\partial_t n_c}{2\sqrt{n_c}} e^{i\phi} - \sqrt{n_c} e^{i\phi} \partial_t \phi. \end{aligned} \quad (4.2)$$

Similarly, by expanding the derivative in r on the right hand side of (4.2), one gets

$$\begin{aligned} &\left(-\frac{\Delta_r}{2} + g[2n_n + n_c] \right) \sqrt{n_c} e^{i\phi} - \frac{i\epsilon}{2\sqrt{n_c}} \Gamma_{12}[f] e^{i\phi} \\ &= e^{i\phi} \left(-\frac{\Delta_r}{2} + g[2n_n + n_c] \right) \sqrt{n_c} - \nabla_r \sqrt{n_c} \nabla_r e^{i\phi} - \frac{1}{2} \sqrt{n_c} \Delta_r e^{i\phi} \\ &\quad - \frac{i\epsilon}{2\sqrt{n_c}} \Gamma_{12}[f] e^{i\phi} \\ &= e^{i\phi} \left(-\frac{\Delta_r}{2} + g[2n_n + n_c] \right) \sqrt{n_c} - i e^{i\phi} \nabla_r \sqrt{n_c} \nabla_r \phi + \frac{1}{2} \sqrt{n_c} |\nabla_r \phi|^2 e^{i\phi} \\ &\quad - i e^{i\phi} \frac{1}{2} \sqrt{n_c} \Delta_r \phi - \frac{i\epsilon}{2\sqrt{n_c}} \Gamma_{12}[f] e^{i\phi}. \end{aligned} \quad (4.3)$$

Combining (4.1), (4.2) and (4.3) and removing the factor $e^{i\phi}$ yield

$$\begin{aligned} &i\hbar \frac{\partial_t n_c}{2\sqrt{n_c}} - \sqrt{n_c} \partial_t \phi \\ &= \left(-\frac{\Delta_r}{2} + g[2n_n + n_c] \right) \sqrt{n_c} - i \nabla_r \sqrt{n_c} \nabla_r \phi + \frac{1}{2} \sqrt{n_c} |\nabla_r \phi|^2 \\ &\quad - i \frac{1}{2} \sqrt{n_c} \Delta_r \phi - \frac{i\epsilon}{2\sqrt{n_c}} \Gamma_{12}[f]. \end{aligned} \quad (4.4)$$

Identifying the imaginary parts on both sides of (4.4), one obtains

$$\frac{\partial_t n_c}{2\sqrt{n_c}} = - \nabla_r \sqrt{n_c} \nabla_r \phi - \frac{1}{2} \sqrt{n_c} \Delta_r \phi - \frac{\epsilon}{2\sqrt{n_c}} \Gamma_{12}[f]. \quad (4.5)$$

Multiplying both sides of the above equation with $2\sqrt{n_c}$ implies

$$\begin{aligned} \partial_t n_c &= - 2\sqrt{n_c} \nabla_r \sqrt{n_c} \nabla_r \phi - n_c \Delta_r \phi - \epsilon \Gamma_{12}[f] \\ &= - \nabla \cdot (n_c v_c) - \epsilon \Gamma_{12}[f]. \end{aligned} \quad (4.6)$$

Let us redefine the condensate velocity as

$$v_c(t, r) = \nabla_r \phi(t, r). \quad (4.7)$$

Identifying the real parts on both sides of (4.4) gives

$$-\sqrt{n_c} \partial_t \phi = \left(-\frac{\Delta_r}{2} + g[2n_n + n_c] \right) \sqrt{n_c} + \frac{1}{2} \sqrt{n_c} |\nabla_r \phi|^2. \quad (4.8)$$

Dividing both sides of the above equation by $\sqrt{n_c}$, and recalling the definition of the condensate chemical potential

$$\mu_c = \frac{1}{\sqrt{n_c}} \left(-\frac{\Delta_r}{2} + g[2n_n + n_c] \right) \sqrt{n_c}, \quad (4.9)$$

one has

$$-\partial_t \phi = \mu_c + \frac{1}{2} |\nabla_r \phi|^2, \quad (4.10)$$

which leads to

$$-\partial_t \nabla_r \phi = \nabla_r \mu_c + \nabla_r \phi \Delta_r \phi. \quad (4.11)$$

Taking into account the definition of the condensate velocity (4.7) gives

$$-\partial_t v_c = \nabla_r \mu_c + v_c \nabla v_c = \nabla_r \mu_c + \frac{\nabla_r v_c^2}{2} \quad (4.12)$$

Combining (4.6) and (4.12), one obtains the following system for the super-fluid of the condensate

$$\begin{aligned} \partial_t n_c + \nabla_r \cdot (n_c v_c) &= -\epsilon \Gamma_{12}[f] \\ \partial_t v_c + \frac{\nabla_r v_c^2}{2} &= -\nabla_r \mu_c. \end{aligned} \quad (4.13)$$

5 The two-fluid Euler quantum hydrodynamic limit

In this section, we will derive a two-fluid Euler quantum hydrodynamic limit from (2.43) - (2.44). In order to obtain the Euler hydrodynamics limit, let us start with the following Hilbert expansion (cf. [15]):

$$f = \sum_{i=0}^n \epsilon^i f^{(i)} + \epsilon^l \zeta, \quad (5.1)$$

in which n and l are positive integers. As a consequence, we can replace f by its Hilbert expansion into

$$\mathcal{D}f = \frac{1}{\epsilon} C_{22}[f],$$

to get a system of equations and a nonlinear equation for the remainder r , which reads as:

$$B_1(f^{(0)}, f^{(0)}) + B_2(f^{(0)}, f^{(0)}, f^{(0)}) = 0, \quad (5.2)$$

$$2B_1(f^{(0)}, f^{(1)}) + 3B_2(f^{(0)}, f^{(0)}, f^{(1)}) = \mathcal{D}f^{(0)}, \quad (5.3)$$

$$\begin{aligned} 2B_1(f^{(0)}, f^{(i)}) + 3B_2(f^{(0)}, f^{(0)}, f^{(i)}) &= \mathcal{D}f^{(i-1)} - \sum_{j=1}^{i-1} B_1(f^{(i)}, f^{(i-j)}) \\ &\quad - \sum_{j,k=1, 0 < j+k < i}^{i-1} B_2(f^{(i)}, f^{(k)}, f^{(i-j-k)}), \end{aligned} \quad (5.4)$$

for $i = 2, 3, \dots, n$.

The equation for the remainder r is as follows:

$$\begin{aligned} \mathcal{D}\varsigma &= \frac{1}{\epsilon} \mathcal{L}\varsigma + 2 \sum_{i=1}^n \epsilon^{i-1} B_1(f^{(i)}, \varsigma) + \epsilon^{l-1} B_1(\varsigma, \varsigma) + 3 \sum_{i=1}^n B_2(\mathcal{F}, f^{(i)}, \varsigma) \\ &\quad + 3 \sum_{i,j=1}^n \epsilon^{i+j-1} B_2(f^{(i)}, f^{(j)}, \varsigma) + 3\epsilon^{(l-1)} B_2(\mathcal{F}, \varsigma, \varsigma) + 3\epsilon^{l-1} \sum_{i=1}^n \epsilon^i B_2(f^{(i)}, \varsigma, \varsigma) \\ &\quad + \epsilon^{2l-1} B_2(\varsigma, \varsigma, \varsigma) + \epsilon^{n-1} \mathfrak{Q}, \end{aligned} \quad (5.5)$$

where \mathfrak{Q} is an operator of $\mathcal{F}, f^{(1)}, \dots, f^{(n)}$.

Let us now consider each equation in the above system. From the first equation (5.2), we deduce that $f^{(0)}$ has to be a Bose-Einstein distribution:

$$f^{(0)} = \mathcal{F}. \quad (5.6)$$

The equations (5.3) and (5.4) lead to linear integral equations for $f^{(1)}, \dots, f^{(i)}$. Thanks to Fredholm's theory, these linear integral equations are solvable if the right hand sides are orthogonal to \mathcal{N} in $L^2(\mathbb{R}^3)$. As a consequence, $f^{(1)}$ can be solved from (5.3), if the following condition is satisfied

$$\mathbb{P}\mathcal{D}\mathcal{F} = 0. \quad (5.7)$$

We recall that \mathbb{P} and $\mathbb{P}^\perp = 1 - \mathbb{P}$ are the orthogonal projection operators onto \mathcal{N} and \mathcal{R} in $L^2(\mathbb{R}^3)$.

5.1 The Euler quantum hydrodynamic limit of the thermal cloud kinetic equation

Computing (5.7) leads to the following system of moment equations for the kinetic equation of the thermal cloud:

$$\begin{aligned} \partial_t n_n + \nabla_r \cdot (n_n v_n) &= 0, \\ n_n (\partial_t + v_n \cdot \nabla) v_{nj} &= - \sum_{i=1}^3 \partial_{r_j} \mathcal{P}_{ij} \\ \partial_t \mathbb{E}_n + \nabla_r \cdot (\mathbb{E}_n v_n) &= - \sum_{i,j=1}^3 \frac{1}{2} (v_{ni} \partial_{r_j} + v_{nj} \partial_{r_i}) \mathcal{P}_{ij}, \end{aligned} \quad (5.8)$$

where

$$\mathcal{P}_{ij} = \int_{\mathbb{R}^3} (p_i - v_{ni}(t, r)) (p_j - v_{nj}(t, r)) \mathcal{F}(t, r, p) dp. \quad (5.9)$$

Observing that

$$\mathcal{P}_{ij}(t, r) = \delta_{ij} \tilde{\mathbb{E}}_n(t, r) = \frac{2}{3} \mathbb{E}_n(t, r), \quad (5.10)$$

one can close the system (5.8) as follows

$$\begin{aligned} \partial_t n_n + \nabla_r \cdot (n_n v_n) &= 0, \\ n_n (\partial_t + v_n \cdot \nabla) v_{nj} &= -\frac{2}{3} \nabla_r \tilde{\mathbb{E}}_n, \\ \partial_t \tilde{\mathbb{E}}_n + \nabla_r \cdot (\tilde{\mathbb{E}}_n v_n) &= -\frac{2}{3} \tilde{\mathbb{E}}_n \nabla_r \cdot v_n - \sum_{i,j=1}^3 \frac{1}{2} (v_{ni} \partial_{rj} + v_{nj} \partial_{ri}) \mathcal{P}_{ij}. \end{aligned} \quad (5.11)$$

If one replaces \mathbb{E}_n by $n_n e$ as in (3.23), the above system becomes

$$\begin{aligned} \partial_t n_n + \nabla_r \cdot (n_n v_n) &= 0, \\ n_n (\partial_t + v_n \cdot \nabla) v_{nj} + \frac{2}{3} \nabla_r (n_n e_n) &= 0 \\ \partial_t e_n + \nabla_r \cdot (e_n v_n) + \frac{2}{3} e_n \nabla_r \cdot v_n &= \frac{1}{n_n} \left[- \sum_{i,j=1}^3 \frac{1}{2} (v_{ni} \partial_{rj} + v_{nj} \partial_{ri}) \mathcal{P}_{ij} \right]. \end{aligned} \quad (5.12)$$

5.2 The Euler two-fluid quantum hydrodynamic limit

Putting the two systems (4.13) and (5.11) together, we get the following two-fluid Euler quantum hydrodynamic limit

$$\begin{aligned} \partial_t n_c + \nabla_r \cdot (n_c v_c) &= 0, \\ \partial_t v_c + \frac{\nabla_r v_c^2}{2} &= -\nabla_r \mu_c, \\ \partial_t n_n + \nabla_r \cdot (n_n v_n) &= 0, \\ n_n (\partial_t + v_n \cdot \nabla) v_{nj} &= -\frac{2}{3} \nabla_r \tilde{\mathbb{E}}_n, \\ \partial_t \tilde{\mathbb{E}}_n + \nabla_r \cdot (\tilde{\mathbb{E}}_n v_n) &= -\frac{2}{3} \tilde{\mathbb{E}}_n \nabla_r \cdot v_n - \sum_{i,j=1}^3 \frac{1}{2} (v_{ni} \partial_{rj} + v_{nj} \partial_{ri}) \mathcal{P}_{ij}, \end{aligned} \quad (5.13)$$

where

$$\mathcal{P}_{ij} = \delta_{ij} \tilde{\mathbb{E}}_n(t, r).$$

In the obtained system (5.13), the condensate is coupled with the non-condensate part through the condensate chemical potential μ_c ; while the non-condensate system is independent of the condensate. The system is, therefore, a weakly coupled one.

Recall that in the above derivation, we assume that $g = \epsilon^{\delta_0}$. If this assumption is replaced by $g = O(1)$, one obtains a very similar system to (5.13), in which the second equation is the same, while the others are modified as follows:

$$\begin{aligned}
\partial_t n_c + \nabla_r \cdot (n_c v_c) &= -\Gamma_{12}[\mathcal{F}], \\
\partial_t n_n + \nabla_r \cdot (n_n v_n) &= \Gamma_{12}[\mathcal{F}], \\
n_n (\partial_t + v_n \cdot \nabla) v_{nj} &= -\frac{2}{3} \nabla_r \tilde{\mathbb{E}}_n - n_n \nabla_r U - m(v_{nj} - v_{cj}) \Gamma_{12}[\mathcal{F}], \\
\partial_t \tilde{\mathbb{E}}_n + \nabla_r \cdot (\tilde{\mathbb{E}}_n v_n) &= -\frac{2}{3} \tilde{\mathbb{E}}_n \nabla_r \cdot v_n - \sum_{i,j=1}^3 \frac{1}{2} (v_{ni} \partial_{rj} + v_{nj} \partial_{ri}) \mathcal{P}_{ij} \\
&\quad + \frac{2}{3} \left(\frac{(v_n - v_c)^2}{2} + \mu_c - U \right) \Gamma_{12}[\mathcal{F}].
\end{aligned} \tag{5.14}$$

Different from (5.13), in (5.14), the condensate and non-condensate parts are coupled through both μ_c and Γ_{12} . Notice that Γ_{12} is already computed in (3.28) and (3.29).

Equivalently, if replacing \mathbb{E}_n by $n_n e$ as in (3.23), one obtains the following equivalent systems of (5.13)

$$\begin{aligned}
\partial_t n_c + \nabla_r \cdot (n_c v_c) &= 0, \\
\partial_t v_c + \frac{\nabla_r v_c^2}{2} &= -\nabla_r \mu_c, \\
\partial_t n_n + \nabla_r \cdot (n_n v_n) &= 0, \\
n_n (\partial_t + v_n \cdot \nabla) v_{nj} + \frac{2}{3} \nabla_r (n_n e_n) &= 0, \\
\partial_t e_n + \nabla_r \cdot (e_n v_n) + \frac{2}{3} e_n \nabla_r \cdot v_n &= \frac{1}{n_n} \left[-\sum_{i,j=1}^3 \frac{1}{2} (v_{ni} \partial_{rj} + v_{nj} \partial_{ri}) \mathcal{P}_{ij} \right],
\end{aligned} \tag{5.15}$$

and of (5.14)

$$\begin{aligned}
\partial_t n_c + \nabla_r \cdot (n_c v_c) &= -\Gamma_{12}[\mathcal{F}], \\
\partial_t n_n + \nabla_r \cdot (n_n v_n) &= \Gamma_{12}[\mathcal{F}], \\
n_n (\partial_t + v_n \cdot \nabla) v_{nj} + \frac{2}{3} \nabla_r (n_n e_n) &= -n_n \nabla_r U - (v_{nj} - v_{cj}) \Gamma_{12}[\mathcal{F}], \\
\partial_t e_n + \nabla_r \cdot (e_n v_n) + \frac{2}{3} e_n \nabla_r \cdot v_n &= \frac{1}{n_n} \left[-\sum_{i,j=1}^3 \frac{1}{2} (v_{ni} \partial_{rj} + v_{nj} \partial_{ri}) \mathcal{P}_{ij} \right. \\
&\quad \left. + \frac{2}{3} \left(\frac{(v_n - v_c)^2}{2} + \mu_c - U + e_n \right) \Gamma_{12}[\mathcal{F}] \right].
\end{aligned} \tag{5.16}$$

6 The two-fluid Navier-Stokes quantum hydrodynamic approximations

This section is devoted to the derivation of the Navier-Stokes approximation of the system (2.43) - (2.44) through the Chapman-Enskog expansion. Similar as in Section 5.1, we also have the expansion:

$$f = \sum_{i=0}^n \epsilon^i f^{(i)} + \epsilon^l \zeta, \quad (6.1)$$

in which n and l are positive integers, and the system (5.2)-(5.4).

Decompose $f^{(i)}$ into two parts

$$f^{(i)} = h^{(i)} + k^{(i)}, \quad (6.2)$$

where

$$h^{(i)} \in \mathcal{R}, \quad k^{(i)} \in \mathcal{N}.$$

From (5.3), one has

$$h^{(1)} = \mathcal{L}^{-1} \mathcal{D} \mathcal{F}. \quad (6.3)$$

Decompose $h^{(1)}$ into the sum of h' and h'' :

$$h^{(1)} = h' + h'',$$

where h' and h'' satisfy the following system of equations:

$$\mathcal{L} h' = \mathbb{P}^\perp \mathcal{D} \mathcal{F}, \quad (6.4)$$

$$\mathbb{P} \mathcal{D} \mathcal{F} = -\epsilon \mathbb{P} \mathcal{D} h', \quad (6.5)$$

$$\mathcal{L} h'' = \epsilon \mathbb{P}^\perp \mathcal{D} k^{(1)}, \quad (6.6)$$

$$\mathbb{P} \mathcal{D} k^{(1)} = -\mathbb{P} \mathcal{D} h'', \quad (6.7)$$

and

$$\mathcal{L} h^{(i)} = \epsilon \mathbb{P}^\perp \mathcal{D} k^{(i)} + \mathbb{P}^\perp \mathcal{D} h^{(i-1)} - \sum_{j=1}^{i-1} Q_1(f^{(j)}, f^{(i-j)}) - \sum_{j,k=0, 0 < j+k < i}^{i-1} Q_2(f^{(j)}, f^{(k)}, f^{(i-j-k)}), \quad (6.8)$$

$$\mathbb{P} \mathcal{D} k^{(i)} = -\mathbb{P} \mathcal{D} h^{(i)}. \quad (6.9)$$

By the Fredholm theory, the system (6.4)-(6.7) can be solved in $L^2(\mathbb{R}^3)$, if

$$\begin{aligned} h' &= \mathcal{L}^{-1}(\mathbb{P}^\perp \mathcal{D} \mathcal{F}), \\ h'' &= \mathcal{L}^{-1}(\epsilon \mathbb{P}^\perp \mathcal{D} k^{(1)}) \end{aligned} \quad (6.10)$$

and

$$\begin{aligned}
h^{(i)} = & \mathcal{L}^{-1} \left(\epsilon \mathbb{P}^\perp \mathcal{D} k^{(i)} + \mathbb{P}^\perp \mathcal{D} h^{(i-1)} - \sum_{j=1}^{i-1} Q_1(f^{(j)}, f^{(i-j)}) \right. \\
& \left. - \sum_{j,k=0; 0 < j+k < i}^{i-1} Q_2(f^{(j)}, f^{(k)}, f^{(i-j-k)}) \right),
\end{aligned} \tag{6.11}$$

for $i = 2, 3, \dots$

The two equations (6.10) lead to the following equation

$$\mathbb{P} \mathcal{D} \mathcal{F} = -\epsilon \mathbb{P} \mathcal{D} \mathcal{L}^{-1} \mathbb{P}^\perp \mathcal{D} \mathcal{F}. \tag{6.12}$$

Equation (6.12) leads to the Navier-Stokes approximation. In Section 6.1, we compute this equation from (6.12). The full Navier-Stokes two-fluid model will be given in Section 6.2.

6.1 Navier-Stokes quantum hydrodynamic approximation of the thermal cloud kinetic equation

In this subsection, we will derive the Navier-Stokes system resulting from (6.12). First, observe that

$$\begin{aligned}
\mathbb{P}^\perp \Pi \mathcal{F} = & \frac{\mathcal{F}^2}{\mathcal{M}} \sum_{i,j=1}^3 \left\{ (p_i - v_{ni})(p_j - v_{nj}) - \frac{1}{3} |p - v_n|^2 \delta_{i,j} \right\} \frac{1}{\tau} \frac{\partial v_{nj}}{\partial x_i} \\
& + \frac{\mathcal{F}^2}{\mathcal{M}} \left\{ |p - v_n|^2 - \frac{10\tau\Omega_2(\gamma)}{3\Omega_1(\gamma)} \right\} \sum_{i=1}^3 (p_i - v_{ni}) \frac{1}{2\tau^2} \frac{\partial \tau}{\partial r_i}.
\end{aligned} \tag{6.13}$$

Classical techniques for the classical Boltzmann collision operator can be applied (cf. [41] - pp. 456-457 and [30, 9]), to get

$$\begin{aligned}
-\mathbb{P} \Pi \mathcal{L}^{-1} \mathbb{P}^\perp \Pi \mathcal{F} = & \frac{\mathcal{F}^2}{\mathcal{M}} \left(\sum_{k=1}^3 \frac{\psi_k}{\omega_k} \left(\sum_{i=1}^3 \frac{\partial}{\partial r_i} \left(\varpi(\gamma, \tau) \left(\frac{\partial v_{nk}}{\partial r_i} + \frac{\partial v_{ni}}{\partial r_k} \right) \right) \right) \right. \\
& - \frac{2}{3} \frac{\partial}{\partial r_k} \left(\varpi(\gamma, \tau) \sum_{i=1}^3 \frac{\partial v_{ni}}{\partial r_i} \right) \Bigg) + 2 \frac{\psi_4}{\omega_4} \left(\sum_{i=1}^3 \frac{\partial}{\partial r_i} \left(\varrho(\gamma, \tau) \frac{\partial \tau}{\partial r_i} \right) \right. \\
& \left. \left. - \frac{2}{3} \varrho(\gamma, \tau) \left(\sum_{i=1}^3 \frac{\partial v_{ni}}{\partial r_i} \right)^2 + \varrho(\gamma, \tau) \sum_{i,k=1}^3 \frac{\partial v_{nk}}{\partial r_i} \left(\frac{\partial v_{nk}}{\partial r_i} + \frac{\partial v_{ni}}{\partial r_k} \right) \right) \right),
\end{aligned} \tag{6.14}$$

where

$$\varpi(\gamma, \tau) = -\frac{1}{\tau} \int_{\mathbb{R}^3} \xi_1 \xi_2 \mathfrak{C}_{12}(\xi) d\xi, \tag{6.15}$$

$$\varrho(\gamma, \tau) = -\frac{1}{4\tau^2} \int_{\mathbb{R}^3} |\xi|^2 \xi_1 \mathfrak{C}_1(\xi) d\xi, \quad (6.16)$$

with ξ_1, x_2 are the components of the vectors $xi = (\xi_1, \xi_2, \xi_3)$ and $\mathfrak{C}_1, \mathfrak{C}_{12}$ are defined in (3.19).

Notice that

$$\mathcal{DF} = \Pi\mathcal{F} + O(\epsilon^{\delta_0}).$$

The first order approximation in terms of ϵ of the quantity $\epsilon \mathbb{P} \mathcal{D} \mathcal{L}^{-1} \mathbb{P}^\perp \mathcal{DF}$ is then $\epsilon \mathbb{P} \Pi \mathcal{L}^{-1} \mathbb{P}^\perp \Pi \mathcal{F}$. The Navier-Stokes system (6.12) becomes

$$\mathbb{P} \mathcal{DF} = -\epsilon \mathbb{P} \Pi \mathcal{L}^{-1} \mathbb{P}^\perp \Pi \mathcal{F}, \quad (6.17)$$

which, thanks to the identity (6.14), leads to

$$\begin{aligned} \partial_t n_n + \nabla_r \cdot (n_n v_n) &= \epsilon \Gamma_{12}[\mathcal{F}], \\ n_n (\partial_t + v_n \cdot \nabla) v_{nj} + \frac{2}{3} \partial_{r_j} (n_n e_n) &= -n_n \nabla_r \epsilon^{\delta_0} U - (v_{nj} - v_{cj}) \epsilon \Gamma_{12}[\mathcal{F}] \\ &+ \epsilon \left[\sum_{i=1}^3 \frac{\partial}{\partial r_i} \left(\bar{\omega}(n_n, e_n) \left(\frac{\partial v_{nj}}{r_i} + \frac{\partial v_{ni}}{r_j} \right) \right) \right. \\ &\quad \left. - \frac{2}{3} \frac{\partial}{\partial r_j} \left(\bar{\omega}(n_n, e_n) \sum_{i=1}^3 \frac{\partial v_{ni}}{r_i} \right) \right], \\ \partial_t e_n + \nabla_r \cdot (e_n v_n) + \frac{2}{3} e_n \nabla_r \cdot v_n &= \frac{1}{n_n} \left[- \sum_{i,j=1}^3 \frac{1}{2} (v_{ni} \partial_{r_j} + v_{nj} \partial_{ri}) \mathcal{P}[\mathcal{F}]_{ij} \right. \\ &\quad \left. + \frac{2}{3} \left(\frac{(v_n - v_c)^2}{2} + \mu_c - \epsilon^{\delta_0} U + e_n \right) \epsilon \Gamma_{12}[\mathcal{F}] \right] \\ &+ \frac{\epsilon}{\mathcal{G}(n_n, e_n)} \left[\sum_{i=1}^3 \frac{\partial}{\partial r_i} \left(\varrho_1(n_n, e_n) \frac{\partial e_n}{\partial r_i} + \varrho_2(n_n, e_n) \frac{\partial n_n}{\partial r_i} \right) \right. \\ &\quad \left. + \bar{\omega}(n_n, e_n) \sum_{i,k=1}^3 \frac{\partial v_{nk}}{\partial x_i} \left(\frac{\partial v_{nk}}{\partial x_i} + \frac{\partial v_{ni}}{\partial x_k} \right) \right. \\ &\quad \left. - \frac{2}{3} \bar{\omega}(n_n, e_n) \left(\sum_{i=1}^3 \frac{\partial v_{ni}}{\partial x_i} \right)^2 \right], \end{aligned} \quad (6.18)$$

where

$$\begin{aligned} \bar{\omega}(n_n, e_n) &= \varpi(\gamma, \tau), \\ \mathcal{G}(n_n, e_n) &= 2^{5/2} \pi \tau^{3/2} \gamma \Omega_1(\gamma), \\ \varrho_1(n_n, e_n) &= \varrho(\gamma, \tau) \frac{\partial \tau}{\partial e_n}, \\ \varrho_2(n_n, e_n) &= \varrho(\gamma, \tau) \frac{\partial \tau}{\partial n_n}. \end{aligned} \quad (6.19)$$

6.2 The Navier-Stokes two-fluid quantum hydrodynamic approximation

Combining (4.13) and (6.18), leads to the following system

$$\begin{aligned}
\partial_t n_c + \nabla_r \cdot (n_c v_c) &= -\epsilon \Gamma_{12}[\mathcal{F}], \\
\partial_t v_c + \frac{\nabla_r v_c^2}{2} &= -\nabla_r \mu_c, \\
\partial_t n_n + \nabla_r \cdot (n_n v_n) &= \Gamma_{12}[\mathcal{F}], \\
n_n (\partial_t + v_n \cdot \nabla) v_{nj} + \frac{2}{3} \partial_{r_j} (n_n e_n) &= -\epsilon^{\delta_0} n_n \nabla_r U - (v_{nj} - v_{cj}) \epsilon \Gamma_{12}[\mathcal{F}] \\
&+ \epsilon \left[\sum_{i=1}^3 \frac{\partial}{\partial r_i} \left(\bar{\omega}(n_n, e_n) \left(\frac{\partial v_{nj}}{r_i} + \frac{\partial v_{ni}}{r_j} \right) \right) \right. \\
&\quad \left. - \frac{2}{3} \frac{\partial}{\partial r_j} \left(\bar{\omega}(n_n, e_n) \sum_{i=1}^3 \frac{\partial v_{ni}}{r_i} \right) \right], \\
\partial_t e_n + \nabla_r \cdot (e_n v_n) + \frac{2}{3} e_n \nabla_r \cdot v_n &= \frac{1}{n_n} \left[- \sum_{i,j=1}^3 \frac{1}{2} (v_{ni} \partial_{r_j} + v_{nj} \partial_{r_i}) \mathcal{P}_{ij} \right. \\
&\quad \left. + \frac{2}{3} \left(\frac{(v_n - v_c)^2}{2} + \mu_c - \epsilon^{\delta_0} U + e_n \right) \epsilon \Gamma_{12}[\mathcal{F}] \right] \\
&+ \frac{\epsilon}{\mathcal{G}(n_n, e_n)} \left[\sum_{i=1}^3 \frac{\partial}{\partial r_i} \left(\varrho_1(n_n, e_n) \frac{\partial e_n}{\partial r_i} + \varrho_2(n_n, e_n) \frac{\partial n_n}{\partial r_i} \right) \right. \\
&\quad \left. + \bar{\omega}(n_n, e_n) \sum_{i,k=1}^3 \frac{\partial v_{nk}}{\partial x_i} \left(\frac{\partial v_{nk}}{\partial x_i} + \frac{\partial v_{ni}}{\partial x_k} \right) \right. \\
&\quad \left. - \frac{2}{3} \bar{\omega}(n_n, e_n) \left(\sum_{i=1}^3 \frac{\partial v_{ni}}{\partial x_i} \right)^2 \right],
\end{aligned} \tag{6.20}$$

where we use the definitions of (6.15), (6.16) and (6.19).

Acknowledgements. This research was supported by NSF grants DMS-1522184 and DMS-1107291: RNMS KI-Net, by NSFC grant No. 91330203, and by the Office of the Vice Chancellor for Research and Graduate Education at the University of WisconsinMadison with funding from the Wisconsin Alumni Research Foundation. M.-B. Tran was supported partially by ERC Advanced Grant DYCON. M.-B. Tran would like to thank Prof. Linda Reichl and Prof. Jay Robert Dorfman for the discussions on the topic.

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